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Scalar Casimir effect in a circular Aharonov–Bohm quantum billiard

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Abstract. This is a study of the Casimir energy associated to a circular quantum billiard threaded by a single line of flux coming from an external magnetic field. Zero-point energies are calculated after applying zeta-function regularization to eigenmode sums and using some recently obtained representations of Bessel zeta functions for negative arguments. The overall flux dependence can be approximated by a quadratic curve.

1. Introduction

Let us consider the quantum mechanical problem of a scalar particle inside a circular Aharonov–Bohm quantum billiard [1–4] of radius a from the viewpoint of field theory. We take a massless field, with a space-dependent part which we call ϕ, and whose eigenmodes ω satisfy the equation (in units such that $\hbar = c = 1$)

$$(-\mathrm{i}\boldsymbol{\nabla} - e\boldsymbol{A})^2 \phi = \omega^2\phi \tag{1.1}$$

where the vector potential $\boldsymbol{A}$ is given by

$$A_r = 0 \qquad A_\varphi = \frac{\Phi}{2\pi r} \tag{1.2}$$

and

$$\alpha = \frac{e\Phi}{2\pi} \tag{1.3}$$

is called the *reduced flux*. Since a billiard is a domain with perfectly reflecting walls, and we imagine an infinitely thin solenoid at the origin—reduced, in $D = 2$, to an unattainable point—the boundary conditions are $\phi = 0$ at $r = 0$ and $r = a$. When there is no flux (free case), the eigenmodes in these circumstances are the zeros of Bessel functions with integer indices l coming from the angular momentum. The solutions for non-zero α have been found in [1, 4], and basically correspond to an index shift with respect to the free case $|l| \to |l - \alpha|$. Later we shall deal with the associated spectrum, but we will first briefly review the zeta function formalism for calculating Casimir energies.

Zero-point energies emerge from mode-sums $\frac{1}{2}\sum_n \omega_n$, and give rise to the celebrated Casimir effect [5–8] (if we were not using the typical QFT units, we should add a factor $\hbar c$ to this sum). Note that the summation extends over all the ω_n's in the set of eigenmodes. As a result, such quantities do usually diverge and call for some regularization to make

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sense of them. To this end, we introduce the usual spectral zeta functions, which will be denoted by

$$\zeta_{\mathcal{M}}(s) = \sum_n \omega_n^{-s} \qquad \zeta_{\mathcal{M}/\mu}(s) = \sum_n \left(\frac{\omega_n}{\mu}\right)^{-s}. \tag{1.4}$$

where μ is an arbitrary scale with mass dimensions, used to work with dimensionless objects. As they stand, these identities hold only for $\mathrm{Re}\, s > s_0$, being s_0 a positive value given by the rightmost pole of $\zeta_{\mathcal{M}}(s)$. However, such a function admits analytic continuation to other values of s, in particular, to negative reals. Then, the finite part of the vacuum energy, E_C, can be found by combining zeta-regularization of the mode-sum and a principal part prescription from [8]:

$$E_C(\mu) = d_f \operatorname*{PP}_{s\to -1}\left[\frac{1}{2}\mu\, \zeta_{\mathcal{M}/\mu}(s)\right] \tag{1.5}$$

where PP denotes principal part and d_f is the number of degrees of freedom associated with the field (in our case, reasoning as in [9], for example, one realizes that $d_f = 2$). Evidently, for this procedure to work we must be able to obtain the analytic continuation of $\zeta_{\mathcal{M}}(s)$ at least to a part of the negative real axis reaching $s = -1$. This point makes our mathematical problem completely different to that in [1–4]; there the aim was the calculation of the ground state energy (and even the next low-lying energies) by the spectral sum method, which only needs values of $\zeta_{\mathcal{M}}(s)$ at positive s's. Now, the analytic continuation of the spectral zeta function to the negative real axis appears as a non-trivial matter.

Since in free or Aharonov–Bohm circular quantum billiards the eigenmodes are zeros of J_ν Bessel functions, we shall introduce the following 'partial-wave' zeta functions for fixed values of ν:

$$\zeta_\nu(s) = \sum_{n=1}^{\infty} j_{\nu,n}^{-s} \qquad \text{for} \quad \mathrm{Re}\, s > 1 \tag{1.6}$$

where $j_{\nu n}$ denotes the nth non-vanishing zero of J_ν (see also [10, 11])†. (Discrete versions of the Bessel problem, their solutions and associated zeta functions have also been studied in [15].)

When considering the whole problem in a D-dimensional space, one must take into account the degeneracy $d(D, l)$ of each angular mode in D dimensions. Therefore, we define the 'complete' spherical zeta function

$$\zeta_{\mathcal{M}}(s) = a^s \sum_{l=l_{\min}}^{\infty} d(D,l) \sum_{n=1}^{\infty} j_{\nu(D,l),n}^{-s} = a^s \sum_{l=l_{\min}}^{\infty} d(D,l)\, \zeta_{\nu(D,l)}(s) \tag{1.7}$$

where $l_{\min}$ is the minimum value of l, $\nu(D, l) = l + D/2 - 1$ and the general form of $d(D, l)$ (see, e.g., [16]) is

$$d(D,l) = (2l + D - 2)\frac{(l + D - 3)!}{l!(D - 2)!}. \tag{1.8}$$

In section 2 we construct these zeta functions for $D = 2$, obtaining their analytic continuation to $s = -1$. The numerical results for the zero-point energy are discussed in section 3. A calculation of a necessary derivative of the Hurwitz zeta function is outlined in the appendix.

† In the mathematical literature, this object taken at even integer s is sometimes called the Rayleigh function [12].

2. The spectral zeta function

2.1. The 'partial-wave' zeta function

Computing the Casimir energy by (1.5) requires the knowledge of the Bessel zeta functions (1.6) at $s = -1$, while the complex domain where (1.6) holds is bounded by $\mathrm{Re}\, s = 1$. This is a serious difficulty, but we know that $\zeta_\nu(s)$ admits an analytic continuation to other values of s. Moreover, in [10, 11] we showed how to obtain an integral representation of this continuation valid for $-1 < \mathrm{Re}\ s < 0$, which reads

$$\zeta_\nu(s) = \frac{s}{\pi} \sin\frac{\pi s}{2} \int_0^\infty \mathrm{d}x\ x^{-s-1} \ln\left[\sqrt{2\pi x}\, \mathrm{e}^{-x} I_\nu(x)\right] \qquad \text{for} \quad -1 < \mathrm{Re}\, s < 0. \tag{2.1}$$

Whenever $\nu \neq 0$ we can work out (2.1) as explained in [13], arriving at

$$\begin{aligned}\zeta_\nu(s) = \frac{1}{4}\sigma_1 \nu^{-s} + \nu^{-s}\frac{s}{\pi}\sin\frac{\pi s}{2}\Bigg[\sigma_2\Bigg\{\frac{1}{2s}B\left(\frac{s+1}{2}, -\frac{s}{2}\right) + 2^{s-1}B\left(\frac{s+1}{2}, -s\right)\\ +2^{s-1}B\left(\frac{s+3}{2}, -s\right)\Bigg\}\nu + \mathcal{S}_N(s,\nu) + \frac{1}{2}\rho B\left(\frac{s+1}{2}, -\frac{s}{2}\right)\frac{1}{\nu}\\ +\overline{\mathcal{J}}_1(s)\frac{1}{\nu} + \sum_{n=2}^{N}\mathcal{J}_n(s)\frac{1}{\nu^n}\Bigg] \qquad \text{with} \quad \sigma_1 = -1 \quad \sigma_2 = 1 \quad \rho = \frac{1}{8}.\end{aligned} \tag{2.2}$$

In addition

$$\begin{aligned}\overline{\mathcal{J}}_1(s) &= -\frac{5}{24}B\left(\frac{s+3}{2}, -\frac{s}{2}\right)\\ \mathcal{J}_n(s) &= \int_0^\infty \mathrm{d}x\ x^{-s-1}\,\mathcal{U}_n(t(x)) \qquad t(x) = \frac{1}{\sqrt{1+x^2}}\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}\mathcal{U}_1(t) &= \tfrac{1}{8}t - \tfrac{5}{24}5t^3\\ \mathcal{U}_2(t) &= \tfrac{1}{16}t^2 - \tfrac{3}{8}t^4 + \tfrac{5}{16}t^6\\ \mathcal{U}_3(t) &= \tfrac{25}{384}t^3 - \tfrac{531}{640}t^5 + \tfrac{221}{128}t^7 - \tfrac{1105}{1152}t^9\\ \mathcal{U}_4(t) &= \tfrac{13}{128}t^4 - \tfrac{71}{32}t^6 + \tfrac{531}{64}t^8 - \tfrac{339}{32}t^{10} + \tfrac{565}{128}t^{12}\\ &\ \vdots\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\mathcal{S}_N(s,\nu) &\equiv \int_0^\infty \mathrm{d}x\ x^{-s-1}\left\{\ln\left[\sqrt{2\pi\nu}(1+x^2)^{1/4}\mathrm{e}^{-\nu\eta(x)}I_\nu(\nu x)\right] - \sum_{n=1}^{N}\frac{\mathcal{U}_n(t(x))}{\nu^n}\right\}\\ \eta(x) &= \sqrt{1+x^2} + \ln\frac{x}{1+\sqrt{1+x^2}}\end{aligned} \tag{2.5}$$

the key point being that $S_N(s,\nu)$ is a *finite* integral at $s = -1$ (The method used in that reference also has similarities to the technique in [14]).

The expressions for the $\mathcal{J}_n(s)$'s are easily obtained from the $\mathcal{U}_n(t)$'s in (2.4). In fact, since

$$\int_0^\infty \mathrm{d}x\ x^{-s-1}\, t(x)^m = \frac{1}{2}B\left(\frac{s+m}{2}, -\frac{s}{2}\right) \tag{2.6}$$

the result of the integration is like making the replacement

$$\begin{aligned} &\mathcal{U}_n(t) \to \mathcal{J}_n(s) \\ &t^m \to \frac{1}{2} B\left(\frac{s+m}{2}, -\frac{s}{2}\right). \end{aligned} \tag{2.7}$$

Expression (2.2) is not valid for $\nu = 0$, since it was obtained from a rescaling $x \to \nu x$ and subsequent application of uniform asymptotic expansions in νx. Furthermore, numerically speaking it is of little use if ν is very small. An alternative representation valid in these conditions is called for. Starting from (2.1), we introduce $1 = \sqrt{x}(1 + x^2)^{1/4}/\sqrt{x}(1 + x^2)^{1/4}$ into the logarithm, separate $\ln(\sqrt{x}/(1 + x^2)^{1/4})$ and integrate, which takes us to

$$\zeta_\nu(s) = -\frac{1}{4} + \frac{s}{\pi} \sin\frac{\pi s}{2} \int_0^\infty \mathrm{d}x\ x^{-s-1} \ln\left[\sqrt{2\pi}(1 + x^2)^{1/4}\,\mathrm{e}^{-x} I_\nu(x)\right]. \tag{2.8}$$

Next, we will subtract and add the asymptotic behaviour of the integrand, which gives rise to a logarithmic divergence on integration. When doing so, we shall write the large-x expansion of $\ln[\cdots]$ as follows:

$$\ln\left[\sqrt{2\pi}(1 + x^2)^{1/4}\,\mathrm{e}^{-x} I_\nu(x)\right] = -\frac{4\nu^2 - 1}{8x} + \mathrm{O}\left(\frac{1}{x^2}\right) = -\frac{4\nu^2 - 1}{8\sqrt{x^2+1}} + \mathrm{O}\left(\frac{1}{x^2+1}\right). \tag{2.9}$$

Thus, the piece we separate can be integrated with the help of (2.6) ($m = 1$ case) and we are left with

$$\begin{aligned} &\zeta_\nu(s) = -\frac{1}{4} + \frac{s}{\pi} \sin\frac{\pi s}{2} \left[\mathcal{T}_\nu(s) - \frac{4\nu^2 - 1}{16} B\left(\frac{s+1}{2}, -\frac{s}{2}\right)\right] \\ &\mathcal{T}_\nu(s) = \int_0^\infty \mathrm{d}x\ x^{-s-1} \left\{\ln\left[\sqrt{2\pi}(1 + x^2)^{1/4}\,\mathrm{e}^{-x} I_\nu(x)\right] + \frac{4\nu^2 - 1}{8\sqrt{x^2+1}}\right\}. \end{aligned} \tag{2.10}$$

Since the above integral is now finite at $s = -1$ we can Laurent-expand without any difficulty about $s = -1$, finding

$$\zeta_\nu(s) = \frac{1 - 4\nu^2}{8\pi} \frac{1}{s+1} + \frac{1 - 4\nu^2}{8\pi}(-1 + \ln 2) - \frac{1}{4} + \frac{1}{\pi}\mathcal{T}_\nu(-1) + \mathrm{O}(s+1). \tag{2.11}$$

In particular, for $\nu = 0$, $\mathcal{T}_0(-1) = 0.7782$ and

$$\zeta_0(s) = \frac{1}{8\pi} \frac{1}{s+1} - 0.0145 + \mathrm{O}(s+1). \tag{2.12}$$

2.1.1. The 'complete' zeta function. Next, we go on to the two-dimensional problem. For the free case in $D = 2$

$$d(2, l) = \begin{cases} d(2, 0) = 1 & \\ d(2, l) = 2 & \text{for } l \neq 0 \end{cases}$$

and $\nu(2, l) = l, l \geqslant 0$. However, as has already been commented, when a magnetic flux line threads the origin the ν's become $|l - \alpha|$'s ([1, 4]). Therefore, the mode sum yields the following complete spectral zeta function

$$\varepsilon^s \zeta_{\mathcal{M}}(s; \alpha) = \sum_{l=-\infty}^{\infty} \zeta_{|l-\alpha|}(s) \tag{2.13}$$

(in our case, $\varepsilon = a^{-1}$). Since this function has the properties

$$\zeta_{\mathcal{M}}(s;\alpha+k) = \zeta_{\mathcal{M}}(s;\alpha) \qquad k \in \mathbb{Z}$$
$$\zeta_{\mathcal{M}}(s;-\alpha) = \zeta_{\mathcal{M}}(s;\alpha) \tag{2.14}$$

(see [4]), it is enough to study it for $0 \leqslant \alpha \leqslant \frac{1}{2}$. Introducing

$$\overline{\zeta_{\mathcal{M}}}(s;\beta) \equiv a^s \sum_{l=0}^{\infty} \zeta_{l+\beta}(s) \tag{2.15}$$

we can write

$$\zeta_{\mathcal{M}}(s;\alpha) = \overline{\zeta_{\mathcal{M}}}(s;\alpha) + \overline{\zeta_{\mathcal{M}}}(s;1-\alpha) \tag{2.16}$$

$$= a^s \zeta_{|\alpha|}(s) + \overline{\zeta_{\mathcal{M}}}(s;1+\alpha) + \overline{\zeta_{\mathcal{M}}}(s;1-\alpha). \tag{2.17}$$

Next we insert expression (2.2) into (2.15) and, using

$$\sum_{l=0}^{\infty} (l+\beta)^{-s} = \zeta_{\mathrm{H}}(s,\beta) \tag{2.18}$$

where ζ_{H} stands for the Hurwitz zeta function, we find

$$\overline{\zeta_{\mathcal{M}}}(s;\beta) = \frac{1}{4}\sigma_1 a^s \zeta_{\mathrm{H}}(s,\beta) + a^s \frac{s}{\pi}\sin\frac{\pi s}{2}\Bigg[\sigma_2\Bigg\{\frac{1}{2s}B\left(\frac{s+1}{2},-\frac{s}{2}\right)$$
$$+2^{s-1}B\left(\frac{s+1}{2},-s\right) + 2^{s-1}B\left(\frac{s+3}{2},-s\right)\Bigg\}\zeta_{\mathrm{H}}(s-1,\beta)$$
$$+\mathcal{S}_N(s,l+\beta)(l+\beta)^{-s} + \frac{1}{2}\rho B\left(\frac{s+1}{2},-\frac{s}{2}\right)\zeta_{\mathrm{H}}(s+1,\beta)$$
$$+\overline{\mathcal{J}}_1(s)\zeta_{\mathrm{H}}(s+1,\beta) + \sum_{n=2}^{N}\mathcal{J}_n(s)\zeta_{\mathrm{H}}(s+n,\beta)\Bigg] \tag{2.19}$$

where the values of σ_1, σ_2 and ρ are those in (2.2). Taking $N = 4$ and Laurent-expanding, this may be written as

$$\overline{\zeta_{\mathcal{M}}}(s;\beta) = \frac{1}{a}\Bigg[-\frac{1}{4}\zeta_{\mathrm{H}}(-1,\beta) + \frac{1}{\pi}\Bigg\{\frac{1}{4}\zeta_{\mathrm{H}}(-2,\beta) - \frac{5}{24}\zeta_{\mathrm{H}}(0,\beta) - \frac{229}{40320}\zeta_{\mathrm{H}}(2,\beta)$$
$$+\frac{35}{65536}\zeta_{\mathrm{H}}(3,\beta) + \sum_{l=0}^{\infty} S_4(-1,l+\beta)(l+\beta)$$
$$+\left(-\frac{\pi}{256} - \frac{1}{2}\zeta_{\mathrm{H}}(-2,\beta) + \frac{1}{8}\zeta_{\mathrm{H}}(0,\beta)\right)\left(\frac{1}{s+1} + \ln a - 1\right)$$
$$-\frac{\pi}{64} + \frac{\ln 2}{16} - \beta\frac{\ln 2}{8} + \frac{\pi\Psi(\beta)}{256} - \left(1 + \frac{1}{2}\ln 2\right)\zeta_{\mathrm{H}}(-2,\beta)$$
$$-\frac{1}{2}\zeta_{\mathrm{H}}'(-2,\beta) + \frac{1}{8}\zeta_{\mathrm{H}}'(0,\beta)\Bigg\} + \mathrm{O}(s+1)\Bigg]. \tag{2.20}$$

Concerning the pole at $s = -1$ of the complete zeta function, by (2.17), (2.11) and (2.20), and noticing that $\zeta_{\mathrm{H}}(-2, 1+\alpha) + \zeta_{\mathrm{H}}(-2, 1-\alpha) = -\alpha^2$, we come to

$$\zeta_{\mathcal{M}}(s;\alpha) = \frac{1}{a}\left[-\frac{1}{128}\frac{1}{s+1} + \mathrm{O}((s+1)^0)\right] \tag{2.21}$$

i.e. the residue is independent of α.

Since we plan to use the same three formulae for calculating the finite parts, it will be necessary to obtain $\zeta'_{\mathrm{H}}(-2,\beta)$ and $\zeta'_{\mathrm{H}}(0,\beta)$ about $\beta = 1$. The second is known (see, e.g., [17]) and amounts to

$$\zeta'_{\mathrm{H}}(0,\beta) = \ln\Gamma(\beta) - \tfrac{1}{2}\ln(2\pi) \tag{2.22}$$

but the first will still give us still some further trouble. Details about its numerical evaluation are supplied in the appendix.

3. Numerical results and comments

We start by the $l = 0$ partial wave zeta-functions obtained from (2.11). Since we are supposing $\alpha \geqslant 0$, the results will be denoted by

$$a^s \zeta_\alpha(s) = \frac{1}{a}\left[r_\alpha\left(\frac{1}{s+1} + \ln a\right) + p_\alpha\right] + \mathrm{O}(s+1) \tag{3.1}$$

where the residues r_α and the finite parts p_α are listed in table 1. The absence of a pole for $\alpha = \frac{1}{2}$ may be regarded as a consequence of the fact that $J_{1/2}(x) \propto \sin x$, and therefore $\zeta_{1/2}(x) = \pi^{-s}\zeta_{\mathrm{R}}(s)$ (ζ_{R} meaning the Riemann zeta function), which is finite at $s = -1$ because $\zeta_{\mathrm{R}}(-1) = -1/12$. Next, we find $\overline{\zeta_{\mathcal{M}}}(s;\beta)$ from (2.20) for the corresponding $\beta = 1 \pm \alpha$'s. We shall employ the notation

$$\overline{\zeta_{\mathcal{M}}}(s;\beta) = \frac{1}{a}\left[\bar{r}_\beta\left(\frac{1}{s+1} + \ln a\right) + \bar{p}_\beta\right] + \mathrm{O}(s+1) \tag{3.2}$$

and list $\bar{r}_\beta$, $\bar{p}_\beta$ in table 2. Now using equation (2.17) and the above results we get

$$\zeta_{\mathcal{M}}(s;\alpha) = \frac{1}{a}\left[-\frac{1}{128}\left(\frac{1}{s+1} + \ln a\right) + q_\alpha\right] + \mathrm{O}(s+1) \tag{3.3}$$

where the α-indepedence of the resdiue has already been explained, and

$$q_\alpha = p_\alpha + \bar{p}_{1+\alpha} + \bar{p}_{1-\alpha}.$$

The values of q_α for different α's between 0 and $\frac{1}{2}$ are given in table 3. By equation (1.5), the zeta-regularized and PP-renormalized Casimir energy is

$$E_C(\mu, a, \alpha) = \frac{1}{a}\left[-\frac{1}{128}\ln(a\mu) + q_\alpha\right]. \tag{3.4}$$

In particular for $\alpha = 0$ one gets the vacuum energy of a *free* scalar field inside the circular domain and satisfying the Dirichlet condition on the boundary, which is

$$E_C(\mu, a, \alpha = 0) = \frac{1}{a}\left[-\frac{1}{128}\ln(a\mu) + 0.0090\right].$$

Table 1. Residues and finite parts of the $l = 0$ partial wave zeta function at $s = -1$.

α	r_α	p_α
0	$1/8\pi = 0.0398$	-0.0145
0.1	$0.12/\pi = 0.0382$	-0.0597
0.2	$0.105/\pi = 0.0334$	-0.1077
0.3	$0.08/\pi = 0.0225$	-0.1578
0.4	$0.045/\pi = 0.0143$	-0.2093
$\frac{1}{2}$	0	$-\pi/12 = -0.2618$

Table 2. Residues and finite parts of the zeta function $\overline{\zeta_{\mathcal{M}}}(s;\beta)$ at $s=-1$.

β	$\bar{r}_\beta$	$\overline{p}_\beta$
$\frac{1}{2}$	$-1/256=-0.0039$	-0.0547
0.6	$-0.0165/\pi-1/256=-0.0091$	-0.0510
0.7	$-0.032/\pi-1/256=-0.0141$	-0.0429
0.8	$-0.0455/\pi-1/256=-0.0184$	-0.0299
0.9	$-0.056/\pi-1/256=-0.0217$	-0.0117
1	$-1/16\pi-1/256=-0.0238$	0.0117
1.1	$-0.064/\pi-1/256=-0.0243$	0.0406
1.2	$-0.0595/\pi-1/256=-0.0228$	0.0749
1.3	$-0.048/\pi-1/256=-0.0192$	0.1146
1.4	$-0.0285/\pi-1/256=-0.0130$	0.1597
$\frac{3}{2}$	$-1/256=-0.0039$	0.2100

Table 3. Finite parts of the complete zeta function at $s=-1$.

α	q_α
0	0.0090
0.1	-0.0308
0.2	-0.0627
0.3	-0.0860
0.4	-0.1006
0.5	-0.1065

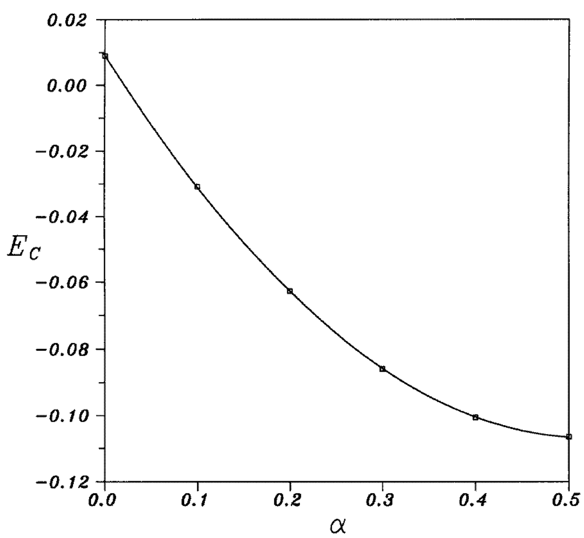

Figure 1. Zero-point energy at $\mu=1/a$ and $a=1$, as a function of the reduced flux α. Clearly, this function is fairly well approximated by a second-degree polynomial (here $0.426\alpha^2-0.444\alpha+0.009$).

Taking $\mu=1/a$, $a=1$, this zero-point energy is plotted in figure 1 as a function of α. Although all this is for $s=-1$, the energy is easily approximated by a second degree polynomial, as happens also with $\zeta_{\mathcal{M}}(s = 2; \alpha)$ (the first reference cited in [1]). The issue of the physical character of this magnitude remains somewhat on an unsatisfactory footing, as equation (3.4) is a scheme-dependent result. However, given that before extracting finite parts all infinities are α-independent, we may conjecture that different renormalizations just fix the origin but the α-dependence remains unchanged. Specifically, in our scheme and in the conditions of figure 1, we realize that the effect of introducing magnetic flux is to lower the zero-point energy of the free case, reversing its sign at $\alpha \simeq 0.02$.

Appendix. s-derivative of the Hurwitz zeta function $\zeta_H(s,\beta)$ at $s=-2$ about $\beta=1$

Here we outline our method for the numerical calculation of

$$\zeta'_H(-2,\beta) = \left.\frac{d}{ds}\zeta_H(s,\beta)\right|_{s=-2}$$

when β is close to 1. For $\beta = 1+\alpha$ (small α) we have $\zeta_H(s,1+\alpha) = \zeta_H(s,\alpha) - \alpha^{-s}$ and therefore

$$\zeta'_H(-2,1+\alpha) = \zeta'_H(-2,\alpha) + \alpha^2 \ln\alpha. \tag{A.1}$$

Let us find $\zeta'_H(-2,\alpha)$ about $\alpha = 0$. First we take

$$\zeta_H(z,\alpha) = \alpha^{-z} + \frac{1}{\Gamma(z)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\alpha^k\Gamma(z+k)\zeta_R(z+k) \tag{A.2}$$

valid when $\mathrm{Re}\, z$ is large enough (ζ_R denotes the Riemann zeta function). We shall continue it back to $z=-2$, being very careful with all the terms containing poles and zeros at this point. Then

$$\begin{aligned}\zeta_H(-2+\epsilon,\alpha) = \alpha^2(1-\epsilon\ln\alpha) + \sum_{k=0}^{2}\frac{2\alpha^k}{k!(2-k)!}\big\{\zeta_R(k-2) + \big[\zeta'_R(k-2)\\ +\zeta_R(k-2)(\psi(3-k)-\psi(3))\big]\epsilon\big\}\\ -\frac{\alpha^3}{3}(1-\psi(3)\epsilon) + 2\epsilon\sum_{k=4}^{\infty}\frac{(-1)^k}{k!}\alpha^k\Gamma(k-2)\zeta_R(k-2) + O(\epsilon^2)\end{aligned} \tag{A.3}$$

and the r.h.s. of (A.1) is given by

$$\begin{aligned}\left.\frac{d}{d\epsilon}\zeta_H(-2+\epsilon,\alpha)\right|_{\epsilon=0} + \alpha^2\ln\alpha\\ = \sum_{k=0}^{2}\frac{2\alpha^k}{k!(2-k)!}\big[\zeta'_R(k-2) + \zeta_R(k-2)(\psi(3-k)-\psi(3))\big]\\ +\frac{\alpha^3}{3}\psi(3) + 2\sum_{k=4}^{\infty}\frac{(-1)^k}{k!}\alpha^k\Gamma(k-2)\zeta_R(k-2).\end{aligned} \tag{A.4}$$

This is the series to be used for the required numerical calculations. When applying it, we will bear in mind that the first two terms contain

$$\begin{aligned}\zeta'_R(-2) &= -\frac{1}{4\pi^2}\zeta_R(3) = -0.030\,448\\ \zeta'_R(-1) &= -0.165\,421\\ \zeta'_R(0) &= -\tfrac{1}{2}\ln(2\pi)\end{aligned} \tag{A.5}$$

where the first result comes from the Riemann zeta function reflexion formula, the second may be found, e.g., in [18] and the third is the $\beta = 1$ case of (2.22).

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